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Difference Equations for Longitudinal Motion in a Synchrotron

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Abstract

A careful but elementary derivation of the difference equations for the longitudinal degree of freedom of particle motion in a synchrotron is given. Very little approximation is required to obtain results which are suitable for numerical calculation and valid under general conditions. With simplifying approximations and specializations, which can be excellent in typical circumstances, the amount of calculation per particle can be reduced sufficiently to make practical simulations with $> 10^9$ particle-turns. The particular approximations used in several versions of the tracking program ESME are identified.

Introduction

There are several available treatments of the equations of motion for synchrotron oscillation, most of them directed toward deriving differential equations or a Hamiltonian from which general properties can be inferred or tractable analytic approximations derived.[1],[2],... It is generally conceded, however, that acceleration in a synchrotron is more accurately expressed by finite difference equations, the difference steps being the azimuthal or particle arrival time separation between localized rf gaps. For purposes of turn-by-turn tracking of particle distributions in longitudinal phasespace, the difference equations are furthermore the more appropriate mathematical framework because they translate directly into numerical operations. This note is motivated by an observation that there is more than one mapping used in tracking programs for longitudinal particle motion;[1],[3],[4] clearly different approximations are used in deriving these maps. When these programs give consistent results, as they generally do, the consistency must result from comparison in a regime where the various approximations are valid. What follows is a careful but elementary derivation of the single-particle equations of motion for the longitudinal degree of freedom of beam particles in a synchrotron. The results are useful as a basis for the simulation of the effect of rf parameter programs or multi-particle dynamics studies.[5] An attempt is made to identify approximations appropriate to various parameter regimes, but the emphasis is on avoiding approximations which limit the range of validity.

A particle circulating in a synchrotron on an orbit of mean radius $R_s = C_s/2\pi$ has average angular velocity

$$\Omega_s = v_s/R_s \tag{1}$$

where v_s is the speed of the particle. Suppose for simplicity that there is a single accelerating gap. If the frequency and amplitude of the rf is set so that the particle receives whatever energy increment is required to keep R_s fixed as the magnitude of the average vertical magnetic field $\langle B_z \rangle$ changes, then the particle is called a synchronous particle, the orbit it follows is the synchronous orbit, and its trajectory is a synchronous trajectory. Imagine looking at the output of a beam current pickup with an oscilloscope that has its time base triggered by the rf system. The signal from the synchronous particle is at a fixed location on the sweep turn after turn. The observed current pulse is the sum from signals of many particles with non-synchronous trajectories. In the conventional operating mode there will be a stable current pulse about the synchronous time over many beam turns; that is, the particle motion is such that trajectories near a synchronous trajectory at one time remain near it for long times. This stability results from so-called "phase focusing" which means that the slope of the rf waveform at the synchronous phase has the same sign as $\frac{d\Omega}{dE}$, the change in circulation frequency with respect to particle energy. For a simple sinusoidal voltage waveform there are two phases per period at which the amplitude yields the correct energy increment. However, the slope of the waveform has opposite sign at these points. The term "synchronous phase" is generally reserved for the stable phase at which the slope of the waveform leads to phase focusing. If the rf system goes through h cycles during the particle circulation period, there are h synchronous trajectories.[1]

Because in the typical circumstance particle trajectories are restricted to the neighborhood of a synchronous trajectory by phase focusing, it is convenient to write the equations for general trajectories in differences of energy and coordinate from a synchronous trajectory so that one has a conventional oscillatory system for the typical case of synchronous acceleration. However, there are other regimes of longitudinal motion of interest like, for example, the perturbed drifting motion in phase displacement acceleration. In this case the synchronous trajectory moves rapidly through the region of longitudinal phasespace occupied by beam particles and may start and end outside the physical aperture. Then the (hypothetical) synchronous particle is quite distinct from the beam particles, and, if the equations of motion are to be useful in

[1] Depending on h and the momentum aperture there may be also $h-1$ faster moving synchronous particles and/or $h+1$ slower moving ones, but this possibility is not usually realized in practice.

this case also, they must not depend on assumptions about small differences between the synchronous trajectory and the trajectories of beam particles. The derivation below retains the idea of synchronous particle and difference coordinates but avoids differential approximations for differences of quantities between the particle trajectory of interest and the synchronous trajectory.

The definition of "synchrotron" generally includes a statement about the constancy of the reference or synchronous trajectory. In what follows this restriction will be considerably looser, allowing small changes in R_s each turn and a synchronous orbit independent of the reference orbit to which the guide field properties are referred. Furthermore, although the derivation is specialized to a single accelerating gap and a difference step of one circulation period to keep the notation simple, the generalizations to multiple gaps and shorter or longer time steps should be clear.

Fundamental Equations in (t,E) Coordinates

Consider the sequence of arrival times of the i-th particle at the one rf gap $t_{i,1}, t_{i,2}, \ldots, t_{i,n}$, where the end of the n-th turn is marked by the n-th crossing of the gap. There is a recursive relation

$$t_{i,n} = t_{i,n-1} + \frac{C_{i,n}}{v_{i,n}} = t_{i,n-1} + \frac{2\pi R_{i,n}}{\beta_{i,n} c} \quad , \tag{2}$$

where $C_{i,n}$ is the length of the orbit for the i-th particle, $R_{i,n}$ is the average radius, and $v_{i,n}$ is the average speed. Define finite differences ΔR and $\Delta\beta$ by

$$\begin{aligned} R_{i,n} &= (1 + \Delta R/R_{s,n}) R_{s,n} & (3) \\ \beta_{i,n} &= (1 + \Delta\beta/\beta_{s,n}) \beta_{s,n} \quad . & (4) \end{aligned}$$

Then

$$t_{i,n} = t_{i,n-1} + \frac{2\pi R_{s,n}}{\beta_{s,n} c} \times \frac{1 + \Delta R/R_{s,n}}{1 + \Delta\beta/\beta_{s,n}} = t_{i,n-1} + \tau_{s,n} S_{i,n} \quad , \tag{5}$$

where $\tau_{s,n}$ is the circulation period for the synchronous particle and the ratio $S_{i,n} = \Omega_{s,n}/\Omega_{i,n}$ may be called the "slip factor". The slip factor is the fraction of a synchronous circulation period which the particle of interest gains or loses with respect to the synchronous particle per turn. At the end of a turn the particle receives an energy increment which depends, of course, on the voltage on the gap at that time

$$E_{i,n} = E_{i,n-1} + eV(\omega_{s,n} t_{i,n}) \quad , \tag{6}$$

where the rf frequency $\omega_{s,n}$ is $h\Omega_{s,n}$ with h integral as required by the definition of a synchronous particle. Substituting for $t_{i,n}$ from eq. 5 one gets

$$E_{i,n} = E_{i,n-1} + eV(\omega_{s,n}[t_{i,n-1} + \tau_{s,n} S_{i,n}]) = E_{i,n-1} + eV(\omega_{s,n} t_{i,n-1} + 2\pi h S_{i,n}) \quad . \tag{7}$$

These equations can be read as a mapping $\mathcal{M}$ of a point $(t_{i,n-1}, E_{i,n-1})$ to another point $(t_{i,n}, E_{i,n})$ in the (t, E) plane. The equations are supposed to represent a conservative process; thus, the mapping should conserve phasespace area. The Jacobian determinant is

$$J(\mathcal{M}) = \frac{\partial(t_{i,n}, E_{i,n})}{\partial(t_{i,n-1}, E_{i,n-1})} = \begin{vmatrix} 1 & \omega_{s,n} eV' \\ \tau_{s,n} \frac{\partial S_{i,n}}{\partial E_{i,n-1}} & 1 + eV' 2\pi h \frac{\partial S_{i,n}}{\partial E_{i,n-1}} \end{vmatrix} \equiv 1 \ . \tag{8}$$

Therefore, phasespace area is conserved and the sequence of $t_{i,n}, E_{i,n}$ for different n lie on a curve of constant H for some Hamiltonian H. The map $\mathcal{M}$ is in the form of an acceleration-free drift in which the coordinate t changes followed by an impulse in which only the conjugate momentum E changes. Forest[6] points out that any map with these properties will be area preserving. It also represents an approximate, but exactly symplectic, integrator of a related continuous Hamiltonian system. In the present case, however, the related system is a continuous approximation to the real impulsive system.

So far an idealized system has been treated essentially exactly. However, the system described is not a very exact model of a synchrotron. Dôme[1] points out that to describe fully the longitudinal dynamics in a synchrotron one must consider the contribution to the force on the particle from the changing magnetic field, *i. e.*, what is generally called betatron acceleration. This force acts continuously around the the machine. It is, however, small compared to the rf force. If one uses an impulsive approximation for the betatron acceleration one makes a small discretization error in a very small term. The actual expression will be written down in a later section, but from the general perspective emphasized here it seems clear that one is closer to reality making a discrete approximation for a small continuous force rather than smoothing the large impulsive rf force over the entire azimuth.

Convenient Coordinates

The difference equations eqs. 5 and 7 are based on comparing the circulation velocity of the i-th particle to the synchronous particle but nonetheless are written in coordinates of integrated circulation time and total energy. It is more natural to consider the difference in arrival times between the i-th and the synchronous particles:

$$\begin{aligned} d_{i,n} &= t_{i,n} - t_{s,n} = t_{i,n} - \sum_{m=1}^{n} \tau_{s,m} = t_{i,n-1} + \tau_{s,n} S_{i,n} - \tau_{s,n} - \sum_{m=1}^{n-1} \tau_{s,m} \\ &= d_{i,n-1} + (S_{i,n} - 1)\tau_{s,n} \ . \end{aligned} \tag{9}$$

The difference $\varepsilon_{i,n}$ between the energy of the i-th particle and the synchronous particle satisfies

$$\begin{aligned}\varepsilon_{i,n} &= \varepsilon_{i,n-1} + eV(\omega_{s,n}[d_{i,n} + t_{s,n}]) - eV(\omega_{s,n}t_{s,n}) \\ &= \varepsilon_{i,n-1} + eV(\omega_{s,n}d_{i,n} + \phi_{s,n}) - eV(\phi_{s,n}) \ , \end{aligned} \tag{10}$$

where $\phi_{s,n}$ is the synchronous phase, *i. e.*, the phase of the rf at the arrival time of the synchronous particle.

The coordinate d is a reasonable one for calculations, but there are two angular variables that may be easier to interpret or incorporate into simulations including other processes like feedback *etc.* The rf phase at the time the particle crosses the gap is simply

$$\phi_{i,n} = \omega_{s,n}t_{i,n} = \omega_{s,n}(d_{i,n} + t_{s,n}) = \varphi_{i,n} + \phi_{s,n} \ , \tag{11}$$

the quantity that appears already in the argument of the potential in the energy equation eq. 10. In these phase-energy coordinates the difference equations become

$$\begin{aligned}\varphi_{i,n} &= \frac{\omega_{s,n}}{\omega_{s,n-1}}\varphi_{i,n-1} + \omega_{s,n}\tau_{s,n}(S_{i,n} - 1) \\ &= \frac{\tau_{s,n-1}}{\tau_{s,n}}\varphi_{i,n-1} + 2\pi h(S_{i,n} - 1)\end{aligned} \tag{12}$$

and

$$\varepsilon_{i,n} = \varepsilon_{i,n-1} + eV(\varphi_{i,n} + \phi_{s,n}) - eV(\phi_{s,n}) \ . \tag{13}$$

The ratio of synchronous periods on successive turns appearing in eq. 12 is very nearly unity and is commonly ignored. It can be neglected without noticeable effect in applications for which R_s is fixed and the range of the synchronous velocity is small, *i. e.*, where the synchronous energy range is small or the synchronous energy is high. If these conditions do not hold, the phasespace area of a distribution will be observed to decrease $\sim \beta_s^{-1}$ under repeated mapping.

In one approach to analyzing the effect of the electromagnetic fields produced by the particle distribution on the motion of individual particles[7] the harmonics of the beam current are calculated from the spacial finite fourier transform of the charge distribution at a fixed time. The FFT is valid for a periodic function; the variable in which the charge distribution is periodic is the cyclic azimuthal variable

$$\Theta_{i,n} = [\Omega_{i,n}d_{i,n} + \pi]_{\mathrm{mod}\,2\pi} - \pi \qquad (-\pi \leq \Theta_{i,n} \leq \pi) \ , \tag{14}$$

where for convenience $\Theta = 0$ is taken as the location of the rf gap. Note that, according to the definition of $S_{i,n}$ and eq. 14, $\Theta_{i,n}$ is less than $\Theta_{i,n-1}$ when the i-th particle is traveling faster than the synchronous particle; thus, the particles travel in the $-\Theta$ sense. This definition is not entirely conventional, but it is convenient. If it is desired to have the particles circulate in the $+\Theta$ sense, equation eq. 14 should include a minus sign. The slip equation eq. 5 becomes

$$\Theta_{i,n} = \frac{\Omega_{i,n}}{\Omega_{i,n-1}}\Theta_{i,n-1} + 2\pi\frac{\Omega_{i,n}}{\Omega_{s,n}}(S_{i,n} - 1) = \frac{\Omega_{i,n}}{\Omega_{i,n-1}}\Theta_{i,n-1} + 2\pi(1 - S_{i,n}^{-1}) \tag{15}$$

and the energy equation becomes

$$\begin{aligned} \varepsilon_{i,n} &= \varepsilon_{i,n-1} + eV(\omega_{s,n}[d_{i,n} + t_{s,n}]) - eV(\omega_{s,n}t_{s,n}) \\ &= \varepsilon_{i,n-1} + eV(hS_{i,n}\Theta_{i,n} + \phi_{s,n}) - eV(\phi_{s,n}) \ . \end{aligned} \tag{16}$$

These equations are quite complicated; the slip factor enters in both, and, worse by far, the Θ equation requires knowing the i-th particle velocity on two successive turns — a significant burden in a large multi-particle calculation. These are the equations that have been used in versions 6, 6.05, 6.5, and 7.0 of the tracking program ESME with the simplifying approximation $\Omega_{i,n}/\Omega_{i,n-1} = 1$ for the Θ equation.[3] The most pronounced effect of this approximation is the β_s^{-1} shrinkage of the phasespace area like that which was described for the similar approximation commonly applied to eq. 12. In version 7.0 this effect was corrected *ad hoc* by introducing the ratio $\tau_{s,n-1}/\tau_{s,n}$ into the energy difference equation. The correction was effective, but it is not consistent with the exact map eqs. 15 and 16.

It would be useful for multi-particle dynamics simulations, where the accuracy may be affected far more by statistical fluctuation in the distribution than by small terms in the single particle dynamics, to have a simpler map to reduce computing time per particle. The time-energy map is very simple; one may try to get a comparably simple angle-energy map by defining a different angular variable which would have the same qualitative interpretation as Θ, that is, would differ from it by an amount negligible for many purposes. If one defines

$$\vartheta_{i,n} = S_{i,n}\Theta_{i,n} \ , \tag{17}$$

the difference equations eqs. 15 and 16 simplify to

$$\vartheta_{i,n} = \frac{\tau_{s,n-1}}{\tau_{s,n}}\vartheta_{i,n-1} + 2\pi(S_{i,n} - 1) \tag{18}$$

$$\varepsilon_{i,n} = \varepsilon_{i,n-1} + eV(h\vartheta_{i,n} + \phi_{s,n}) - eV(\phi_{s,n}) \ . \tag{19}$$

By comparing this map to eqs. 12 and 13 one sees that

$$\vartheta_{i,n} = \varphi_{i,n}/h \ . \tag{20}$$

Thus, one has a nearly circular development. However, a little something has been exposed in the process. From it one sees explicitly the generally ignored distinction between the azimuthal coordinate and the phase coordinate for beam particles.

Clearly, to know the phase the rf will have when a particular particle reaches the gap one must know not only the azimuth of the particle but also its velocity. The velocity information is contained in the slip factor $S_{i,n}$; setting it to one is making the approximation that all particles have the synchronous velocity. On the other hand, it is true that for many purposes the distinction can be ignored because

$$\begin{aligned} S_{i,n} &= \frac{1+\Delta R/R_{s,n}}{1+\Delta\beta/\beta_{s,n}} = \frac{1+\alpha_p \Delta p/p_{s,n}}{1+\gamma_{s,n}^{-2}\Delta p/p_{s,n}} \\ &\approx 1+\eta\frac{\Delta p}{p_{s,n}} \ , \end{aligned} \tag{21}$$

where as usual $\eta = \gamma_T^{-2} - \gamma_{s,n}^{-2}$. In the present version of ESME (version 7.1)[8] the choice has been made to use the precise map eqs. 18 and 19 but not to distinguish between ϑ and Θ when constructing the fourier series for the charge distribution. This approximation is the same as the one used in converting the charge distribution into a current distribution by multiplying throughout by v_s. For the transient analysis of the interaction of the beam current with a high-Q resonator,[3] the $\vartheta_{i,n}$ (or $d_{i,n}$) distribution is exactly what is wanted. In versions of ESME before v. 6, eqs. 18 and 19 were used with the approximation of $\tau_{s,n-1}/\tau_{s,n} = 1$.[9]

If one applies the approximation eq. 21 to the difference equations eqs. 12 and 13 or 18 and 19 and in addition approximates $\tau_{s,n-1}/\tau_{s,n}$ by one, all non-linear dependence on the coordinates is removed except for that in the potential, resulting in a simple mapping which can be used to advantage in large scale tracking calculations or even with a hand calculator if desired:

$$\varphi_{i,n} = \varphi_{i,n-1} + \frac{2\pi h\eta}{\beta_s^2 E_s}\varepsilon_{i,n-1} \tag{22}$$

$$\varepsilon_{i,n} = \varepsilon_{i,n-1} + eV(\varphi_{i,n}+\phi_{s,n}) - eV(\phi_{s,n}) \ . \tag{23}$$

This simple pair is adequate for a large majority of typical tracking calculations where the basic purpose is to study effects of different potentials. The kinds of situation where one should be wary of using it uncritically include near transition energy where first order terms in $S_{i,n}$ cancel, at low energies where the range of velocities in the distribution is more significant, with atypical distributions having large momentum spread or mean energy far from E_s, and in long multi-turn simulations where β_s changes appreciably.

Probably it has not escaped the reader's notice that in going from the (t, E) map eqs. 5 and 6 to the (φ, ε) map eqs. 12 and 13 the coordinates are no longer a canonical pair. The equations are still valid of course, but one must convert the $\Delta\varphi - \Delta E$ phasespace area to energy-time units to compare results at different E_s. Because the $\varphi - E$ units are very physical and it is straightforward to convert the phasespace area, the author has always used ε with the Θ, ϑ, or φ variables for application to tracking calculations. However, for the purposes of analysis or further development of the equations of motion it may be useful to have an area-preserving map. With the substitutions

$$\begin{aligned} \varphi_{i,n} &= \omega_{s,n} d_{i,n} & (24) \\ e_{i,n} &= \varepsilon_{i,n}/\omega_{s,n} & (25) \end{aligned}$$

into the eqs. 9 and 10 one arrives at the map $\mathbf{M}$:

$$\begin{aligned} \varphi_{i,n} &= \frac{\omega_{s,n}}{\omega_{s,n-1}}\varphi_{i,n-1} + 2\pi h(S_{i,n} - 1) & (26) \\ e_{i,n} &= \frac{\omega_{s,n-1}}{\omega_{s,n}} e_{i,n-1} + \frac{e}{\omega_{s,n}}[V(\varphi_{i,n} + \phi_{s,n}) - V(\phi_{s,n})] \ . & (27) \end{aligned}$$

The Jacobian is

$$J(\mathbf{M}) = \frac{\partial(\varphi_{i,n}, e_{i,n})}{\partial(\varphi_{i,n-1}, e_{i,n-1})} = \begin{vmatrix} \frac{\omega_{s,n}}{\omega_{s,n-1}} & \frac{e}{\omega_{s,n}} eV' \\ 2\pi h \frac{\partial S_{i,n}}{\partial e_{i,n-1}} & \frac{\omega_{s,n-1}}{\omega_{s,n}} + \frac{e}{\omega_{s,n}} V' 2\pi h \frac{\partial S_{i,n}}{\partial e_{i,n-1}} \end{vmatrix} \equiv 1 \qquad (28)$$

as desired.

Other Basics

The principal concern in this note has been to write down the basic map in a precise form. However, there are other matters that may be fundamental to accurate modeling of a real synchrotron. In this section the effect of several cavities, longitudinal spacecharge force, and betatron acceleration are considered. The last of these has conceptual importance even if it is to be ignored in practice because considerable emphasis has been placed on precision in the basic map, but it does not include an effect which is present whenever there is changing magnetic field.

If N cavities are equally spaced about the ring, have the same voltage, and are phased to have the same phase for the synchronous particle, the effect is represented by mapping each turn with N applications of the map with $1/N$ of the total voltage and $1/N$ of the phase slip in each iteration. One can see that in principle it is possible to treat an arbitrary spacing of cavities with differing dispersion between them by representing each of the sections with different coefficients in the difference equations. Simply because the accelerator being modeled has more than one rf gap does not mean, of course, that one should necessarily use multiple maps per turn. The phase slip between gaps is usually very small so that the approximation of one per turn is generally excellent. Because the differential equation, which in some sense corresponds to an infinite number of gaps per turn, generally gives results consistent with the one-turn map, one will map more than once per turn only in unusual circumstances. The small amplitude synchrotron tune is a measure of the phase slip per turn; if it exceeds $O(10^{-2})$, a mapping step of one turn is rather coarse, and it could be important to represent more accurately the actual distribution of the cavities.

Just as the circumference can be subdivided by multiple cavities, the drift can be broken into as many segments as the adequate discrete approximation to any continuously distributed force may require. In particular, when one evaluates the effect of beam spacecharge in high intensity and/or low energy accelerators, it may not be adequate to approximate the force with a single kick per turn. Koscielniak has found[4] spurious clumping and breakup of the distribution when the integration step for the spacecharge force is too large. However, the drift equation can be applied separately for each of the requisite number of segments with spacecharge kicks interspersed. When going to small fractions of a turn it may be efficient to apply the small correction $\tau_{s,n-1}/\tau_{s,n}$ only at the end of a turn or perhaps at the end of an inter-cavity segment. Techniques for finding the spacecharge forces from the particle distribution are discussed elsewhere.[3],[7]

The betatron acceleration per turn is

$$\delta E^{(\beta)} = -e \oint \dot{B}_z r \, d\theta \, dr \quad , \tag{29}$$

where the polar cylindrical coordinates have their origin at the center of the ring and the integration over r includes all of the area inside the orbit in which there is changing flux. The change in total energy is very small in most synchrotrons, and, as far as the synchronous particle is concerned, its effect is simply to reduce slightly the magnitude of the synchronous phase. That is, the zero-th order effect of the betatron acceleration is accounted for by, so to speak, renormalizing the synchronous phase so that rf acceleration substitutes for the missing betatron contribution. However, there is a differential acceleration which changes the equations of motion in a more fundamental way; *viz.*, particles with higher momentum are accelerated more than particles with lower momentum independently of their phase because the higher momentum orbit always encloses more flux. Assume that that $\phi_{s,n}$ is adjusted to account for the acceleration of the synchronous particle. The difference in betatron acceleration for the i-th particle and the synchronous particle is

$$\begin{aligned} \Delta E^{(\beta)}_{i,n} &= -e \oint \dot{B}_z (r_{i,n} - r_{s,n}) \, dr \, d\theta \\ &\approx -e \langle \dot{B}_z \rangle (R_{i,n} - R_{s,n}) 2\pi R_{s,n} \quad . \end{aligned} \tag{30}$$

Because

$$p_s = -e \langle B_z \rangle R_s \tag{31}$$

one can write

$$\Delta E^{(\beta)}_{i,n} = 2\pi \dot{p}_{s,n}(R_{i,n} - R_{s,n}) \quad . \tag{32}$$

Because the synchronous phase has been adjusted to include the betatron acceleration of the synchronous particle

$$2\pi R_{s,n}\dot{p}_{s,n} \equiv eV(\phi_{s,n}) \quad . \tag{33}$$

Thus, by replacing $R_{i,n} - R_{s,n}$ with $\alpha_p R_{s,n}\Delta p/p_{s,n}$, one can write

$$\Delta E^{(\beta)}_{i,n} = 2\pi(\alpha_p R_{s,n}\frac{\Delta p}{p_{s,n}})\dot{p}_{s,n} = eV(\phi_{s,n})\frac{\alpha_p \varepsilon_{i,n-1}}{\beta^2_{s,n}E_{s,n}} \quad . \tag{34}$$

The betatron acceleration term is small, but not obviously negligible with respect to kinematic terms that received careful attention above. Nonetheless, the approximations made in evaluating $\Delta E^{(\beta)}_{i,n}$ are more than adequate for synchrotrons. By the convention that the rf kick marks the end of a turn one has for the augmented energy equation

$$\varepsilon_{i,n} = \varepsilon_{i,n-1} + eV(h\vartheta_{i,n} + \phi_{s,n}) - \left(1 - \frac{\alpha_p \varepsilon_{i,n-1}}{\beta^2_{s,n}E_{s,n}}\right) eV(\phi_{s,n}) \quad . \tag{35}$$

However, there is an argument based on the differential equations of longitudinal motion[1] which shows the added term less important than one might assume from eq. 35. If the rf acceleration is smoothed over the entire circumference, the differential betatron acceleration between the synchronous and the other particles can also be represented by changing $\phi_{s,n}$ by a small amount, but there appears to be no analogous argument that one can frame using the difference equations only. So, in principle, and conceivably in practice for special circumstances, one should keep in mind the possibility of significant effect from betatron acceleration. However, if the acceleration is sufficiently gentle that the single-turn map and the differential equation give effectively identical results, one is always justified in neglecting an explicit betatron contribution.

Conclusion

The foregoing discussion of the finite difference equations for single particle motion in longitudinal phasespace coordinates has been aimed at identifying appropriate maps for use in particle tracking programs for studies of both rf parameter programs and multi-particle beam dynamics. The choice that has been made for ESME (v. 7.1) is the set of eqs. 18 and 19 which ignores what has been called the differential betatron acceleration entirely. The program also ignores the distinction between the truly azimuthal variable Θ and the phase related variable ϑ. This differs from a choice in earlier versions that employed Θ but took the ratio $\Omega_{i,n}/\Omega_{i,n-1}$ to be identically one. The most noticeable effect of that approximation is that the longitudinal emittance shrank with increasing particle velocity. This presented no problem for the Fermilab accelerators for which the code was being employed, but certainly can be noticeable in many-turn tracking in lower energy accelerators. Figure 1 shows the the rms emittance of an 0.15 eVs bunch accelerated from 2 MeV to 208 MeV in a machine with the basic parameters of the Loma Linda medical accelerator. The tracking is for 2.33×10^6 turns using the difference equations of version 7.1 of ESME. Figure 2 shows a comparable run using version 6.5. The fluctuations in emittance result from the statistical fluctuation produced using a roughly matched initial distribution of only 143 particles. The initial distribution was enclosed in a matched contour but the density was not constructed to reflect the dependence of the particle velocity on phasespace coordinates.[2]

Acknowledgements

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[2]An approach to generating a distribution with a density that reflects the Hamiltonian flow for a sinusoidal potential is given in a TRIUMF Design Note by Koscielniak.[10] When the potential is complicated the straightforward way to construct a matched distribution is by adiabatic capture.

References

[1] G. Dôme, "Theory of RF Acceleration and RF Noise", in proc. of the CERN Accelerator School Antiprotons for Colliding Beam Facilities, pp215–238, CERN 84-15(20 December 1984)

[2] W. T. Weng, "Longitudinal Motion", in Physics of Particle Accelerators, AIP Conf. Proc. 184, pp 243–287, Am. Inst. of Phys., New York (1989)

[3] J. A. MacLachlan, "Fundamentals of Particle Tracking for the Longitudinal Projection of Beam Phasespace in Synchrotrons", Fermilab FN-481, (15 April 1988), unpublished

[4] Shane R. Koscielniak, "The LONGID Simulation Code", Proc. of the European Part. Acc. Conf., held in Rome (7–11 June 1988)

[5] J. A. MacLachlan, "Particle Tracking in E - ϕ Space as a Design Tool for Cyclic Accelerators", Proc. 1987 IEEE Particle Accelerator Conference held in Washington, D.C., pp1087–1089 (16 – 19 March 1987)

[6] Etienne Forest, "Canonical Integrators as Tracking Codes (or How to Integrate Perturbation Theory with Tracking)", SSC-138 (September 1987), unpublished

[7] J. A. MacLachlan, "Longitudinal Phasespace Tracking with Spacecharge and Wall Coupling Impedance", Fermilab FN-446 (February 1987), unpublished

[8] S. Stahl and J. A. MacLachlan, "User's Guide for ESME (v. 7.1)" near release as Fermilab TM

[9] J. A. MacLachlan, "ESME: Longitudinal Phasespace Particle Tracking — Program Documentation", Fermilab TM-1274(May 1984), unpublished (largely obsolete)

[10] Shane Koscielniak, "Note on Generating Starting Ensembles", TRIUMF Design Note TRI-DN-89-29(October 1989), unpublished

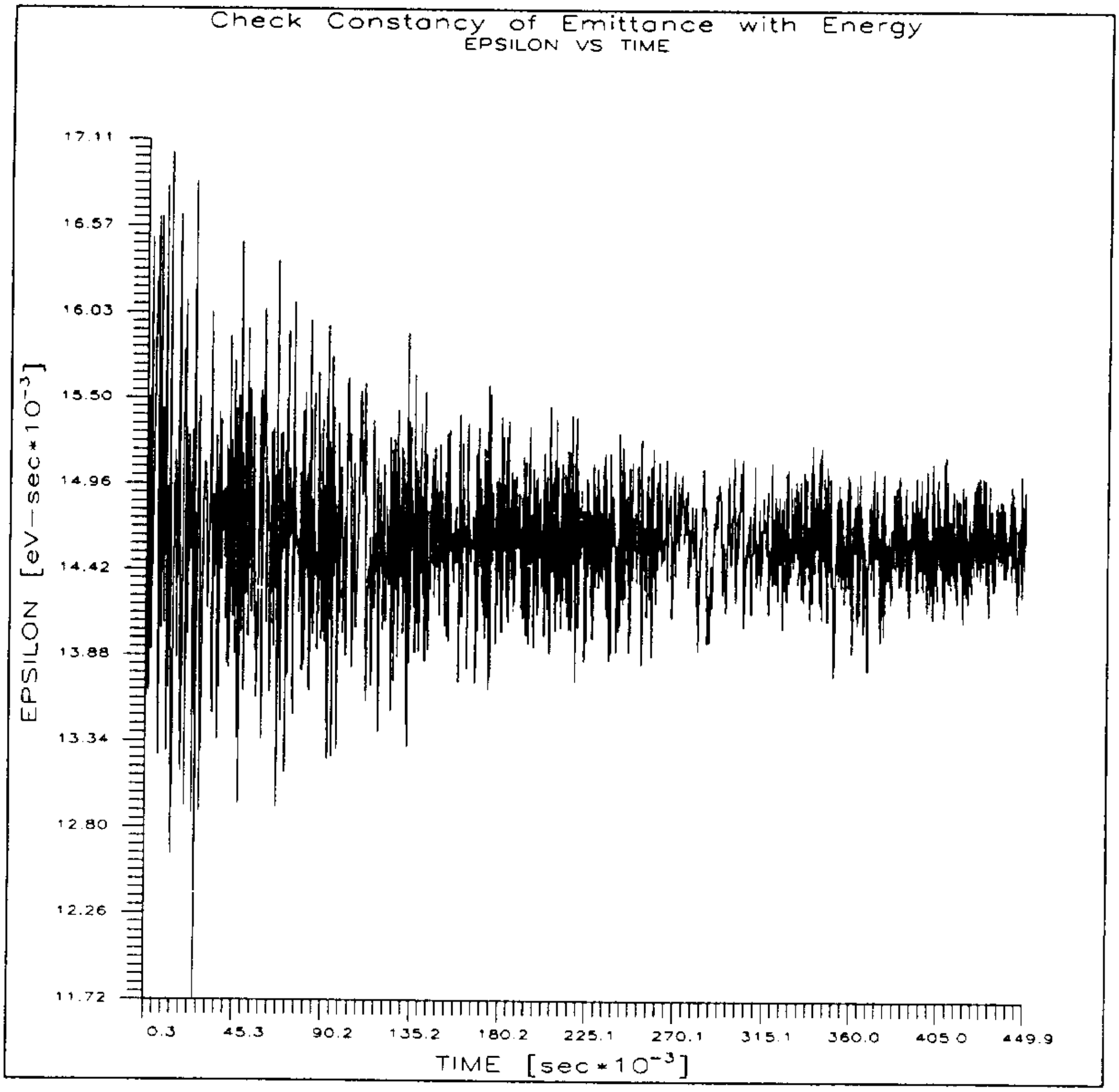

Figure 1: RMS emittance tracked from 2 to 208 MeV for 143 particles using eqs. 18 and 19 in v. 7.1 of ESME

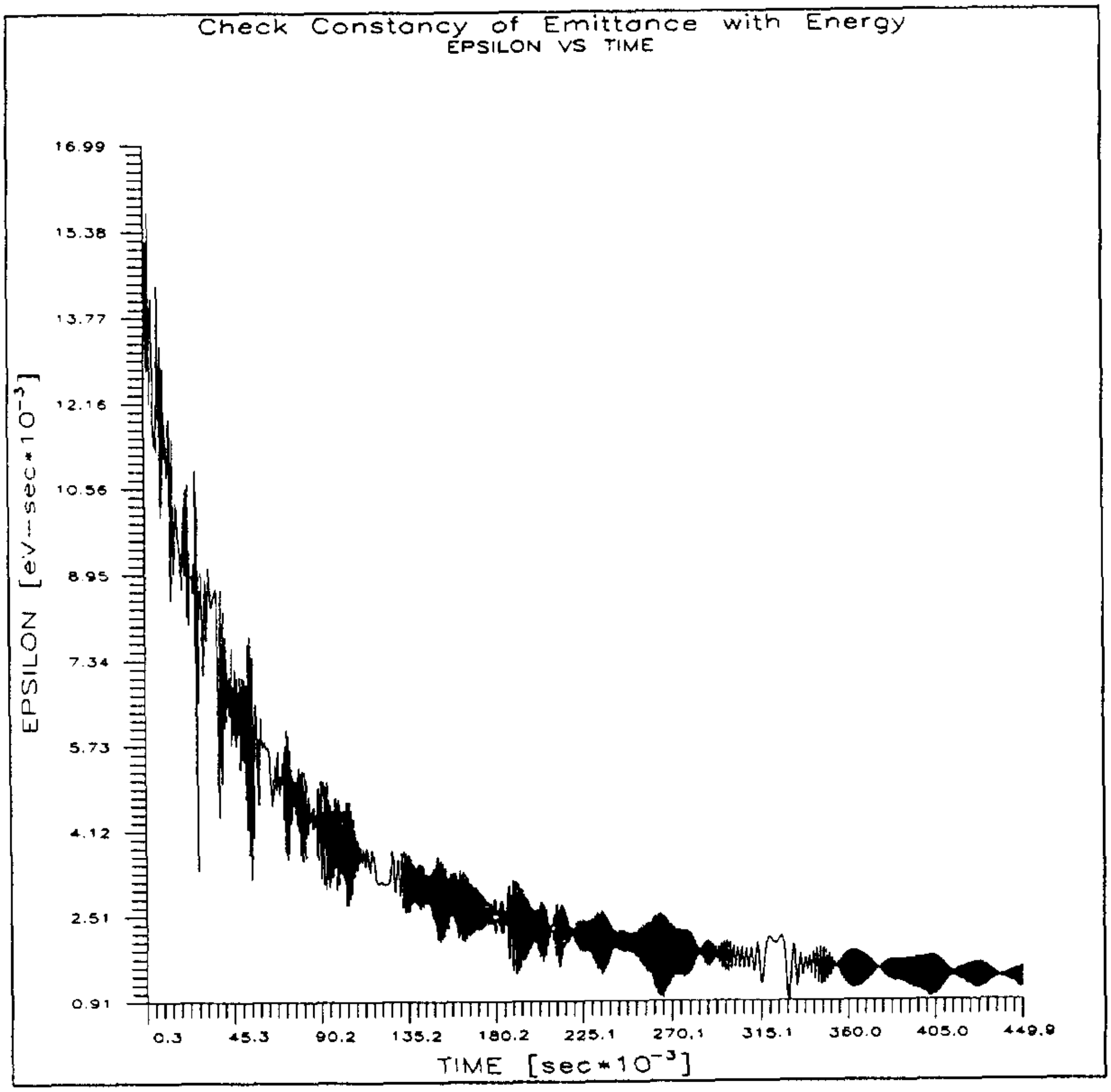

Figure 2: RMS emittance tracked from 2 to 208 MeV for 143 particles using eqs. 15 and 16 in v. 6.5 of ESME